

THE COCYCLE STRUCTURE OF THE ALEXANDER f -QUANDLES ON FINITE FIELDS

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ABSTRACT. We determine the second, third, and fourth cohomology groups of Alexander f -quandles of the form $\mathbb{F}_q[T, S]/(T - \omega, S - \beta)$, where \mathbb{F}_q denotes the finite field of order q , $\omega \in \mathbb{F}_q \setminus \{0, 1\}$, and $\beta \in \mathbb{F}_q$.

1. INTRODUCTION

Quandles are in general non-associative structures whose axioms correspond to the algebraic distillation of the three Reidemeister moves in knot theory. They were introduced independently in the 1980s by Joyce [12] and Matveev [15]. Quandles were used to construct representations of the braid groups. Thus giving constructions of invariants of knots and knotted surfaces as can be seen in [3, 4, 5]. They have been also investigated in the topological context [7, 22] and also for their own right as other non-associative algebraic structures [9, 10, 11, 23]. For more details and recent account on quandles see [8, 18].

Motivated by Hom-algebra structures [14], f -racks, f -quandles and their cohomology theory were introduced and investigated in [6]. Explicit cocycles of this quandle cohomology may be used in the study of Knot Theory, thus in this paper, we investigate the second, third, and fourth cohomology groups of Alexander f -quandles [6]. Our work is motivated by [16], [17], and [20]. Precisely we give basis for the cohomology group $H^n((X, *, f); \mathbb{F}_q)$ with $n = 2, 3$ and 4 .

Through out this paper, let p be a prime, $q = p^m$, and \mathbb{F}_q denote the finite field of order q . Let $M = \mathbb{Z}[\omega^\pm, \beta] = \mathbb{F}_q$, where $\omega (\neq 1)$ and β be non-zero elements of \mathbb{F}_q . Let k be an algebraic closure of \mathbb{F}_q . For $n = 2, 3, 4$, we wish to calculate the Cohomology $H^n(\mathbb{F}_q[T, S]/(T - \omega, S - \beta), k)$ of the Alexander f -quandle $\mathbb{F}_q[T, S]/(T - \omega, S - \beta)$ with coefficients in k .

At the end of each of sections 3, 4 and 5, we provide basis for 2-cocycle, 3-cocycle and 4-cocycle in theorems 3.2, 4.10 and 5.9 respectively. The proofs of this theorems are similar to that of [16]. These proofs will appear in future work.

The paper is organized as follows. In Section 2, we present some preliminaries that will be used throughout the paper. In Sections 3, 4, and 5, we survey 2-cocycles, 3-cocycles, and 4-cocycles of Alexander f -quandles, respectively. We also give some examples in each section.

2. PRELIMINARIES

In this section, we list some preliminaries that will be useful in latter sections.

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Definition 2.1. ([6, Definition 2.1]) An f -quandle is a set X equipped with a binary operation $*$: $X \times X \rightarrow X$ and a map $f : X \rightarrow X$ satisfying the following conditions:

For each $x \in X$, the identity

$$(2.1) \quad x * x = f(x)$$

holds. For any $x, y \in X$, there exists a unique $z \in X$ such that

$$(2.2) \quad z * y = f(x).$$

$$(2.3) \quad (x * y) * f(z) = (x * z) * (y * z)$$

We denote f -quandle by $(X, *, f)$.

Any $\mathbb{Z}[\omega^\pm, \beta]$ -module M is an f -quandle with

$$x * y = \omega \cdot x + \beta \cdot y$$

for $x, y \in M$ with $\omega\beta = \beta\omega$, and we call it an *Alexander f -quandle* ([6, Example 2.1 item (4)]).

Remark 2.2. When f is the identity map and $\beta = 1 - \omega$ above, then $(X, *)$ is a quandle and $(M, *)$ is an Alexander quandle as usual.

Theorem 2.3. ([6, Theorem 5.1]) Let $(X, *, f)$ be a f -quandle, f be a quandle morphism and A be an abelian group. The following family of operators $\delta^n : C^n(X) \rightarrow C^{n+1}(X)$ defines a cohomology complex $C^*(X, *, f, A)$.

$$\begin{aligned} \delta^n \phi(x_1, \dots, x_{n+1}) &= (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i \eta_{[x_1, \dots, \hat{x}_i, \dots, x_{n+1}], f^{\{i-2\}}[x_i, \dots, x_{n+1}]} \phi(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \\ &\quad - (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i \phi(x_1 * x_i, x_2 * x_i, \dots, x_{i-1} * x_i, f(x_{i+1}), \dots, f(x_{n+1})) \\ &\quad + (-1)^{n+1} \tau_{[x_1, x_3, \dots, x_{n+1}], [x_2, \dots, x_{n+1}]} \phi(x_2, \dots, x_{n+1}), \end{aligned}$$

where $[x_1, x_2, x_3, x_4, \dots, x_n] = ((\dots (x_1 * x_2) * f(x_3)) * f^2(x_4)) * \dots) * f^{n-2}(x_n)$. Note that for $i < n$, we have

$$[x_1, x_2, x_3, x_4, \dots, x_n] = [x_1, \dots, \hat{x}_i, \dots, x_n] * f^{i-2}[x_i, \dots, x_n]$$

As in the standard quandle cohomology theory, the degenerate subcomplex is given by $C_n^D = \{(x_1, x_2, \dots, x_n) \in X^n; x_i = x_{i+1} \text{ for } i \geq 2\}$. A similar degenerate subcomplex appeared in [19] under the name of *late degenerate quandles*.

Under the assumption that $\eta = id$ and $\tau = 0$, we can re-write the cohomology complex in Theorem 2.3 as follows.

$$\begin{aligned}
(2.4) \quad & \delta^n \phi(x_1, \dots, x_{n+1}) \\
&= (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i \phi(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \\
&\quad - (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i \phi(x_1 * x_i, x_2 * x_i, \dots, x_{i-1} * x_i, f(x_{i+1}), \dots, f(x_{n+1})).
\end{aligned}$$

We will reformulate the f -quandle cohomology for convenient of calculations.

Let $U_1 = x_1 - x_2, U_2 = x_2 - x_3, \dots, U_i = x_i - x_{i+1}, \dots, U_n = x_n - x_{n+1}, U_{n+1} = x_{n+1}$ for $i = 1, 2, \dots, n$.

Then (2.4) becomes

$$\begin{aligned}
(2.5) \quad & \delta^n \phi(U_1, \dots, U_{n+1}) \\
&= (-1)^{n+1} \sum_{i=1}^n (-1)^i \phi(U_1, \dots, U_{i-1}, U_i + U_{i+1}, U_{i+2}, \dots, U_{n+1}) \\
&\quad - (-1)^{n+1} \sum_{i=1}^n (-1)^i \phi(\omega U_1, \omega U_2, \dots, \omega U_{i-1}, \omega U_i + (\omega + \beta) U_{i+1}, f(U_{i+2}), \dots, f(U_{n+1}))
\end{aligned}$$

The following formula is a generalization of [20, Eq. (3)] when $\eta = id$ and $\tau = 0$ with

$C_d^n(X) := \{\sum a_{i_1, \dots, i_n} \cdot U_1^{i_1} \dots U_n^{i_n} \in C^n(X) \mid \sum_{1 \leq k \leq n} i_k = d\}$ and $\text{degree}(f_a) = d_a$.

$$\begin{aligned}
(2.6) \quad & \delta_n(f)(U_1, \dots, U_n, U_{n+1}) = \sum_{0 \leq a \leq p-1} \delta_{n-1}(f_a)(U_1, \dots, U_n) \cdot U_{n+1}^a \\
&+ (-1)^{n-1} \sum_{0 \leq a \leq p-1} f_a(U_1, \dots, U_{n-1})(U_n + U_{n+1})^a \\
&\quad - (-1)^{n-1} \sum_{0 \leq a \leq p-1} f_a(U_1, \dots, U_{n-1}) \omega^{d_a} (\omega + \beta)^{d-d_a-a} (\omega U_n + (\omega + \beta) U_{n+1})^a.
\end{aligned}$$

3. THE 2-COCYCLES

In this section, we investigate the 2-cocycles. Precisely we provide basis of the second cohomology $H_Q^2((X, *, f); \mathbb{F}_q)$.

Proposition 3.1. *If $\omega^{p^t+p^s} = 1$ and $(\omega + \beta)^{p^t+p^s} = 1$, where s and t are non-negative integers, then $U_1^{p^t} U_2^{p^s}$ is a 2-cocycle.*

Proof. By (2.5), we have

$$\delta(U_1^{p^t}) = (U_1 + U_2)^{p^t} - (\omega U_1 + (\omega + \beta) U_2)^{p^t}.$$

Then it follows from (2.5) and (2.6) that

$$(3.1) \quad \begin{aligned} \delta(U_1^{p^t} U_2^{p^s}) &= \delta(U_1^{p^t}) U_3^{p^s} - U_1^{p^t} (U_2 + U_3)^{p^s} \\ &\quad + U_1^{p^t} \omega^{d_a} (\omega + \beta)^{d-d_a-a} (\omega U_2 + (\omega + \beta) U_3)^{p^s}. \end{aligned}$$

Also, note that $d_a = p^t$, $a = p^s$ and $d = p^t + p^s$. Then we have from (3.1)

$$(3.2) \quad \begin{aligned} \delta(U_1^{p^t} U_2^{p^s}) &= (U_1 + U_2)^{p^t} U_3^{p^s} - (\omega + \beta)^{p^s} (\omega U_1 + (\omega + \beta) U_2)^{p^t} U_3^{p^s} - U_1^{p^t} (U_2 + U_3)^{p^s} \\ &\quad + U_1^{p^t} \omega^{p^t} (\omega + \beta)^0 (\omega U_2 + (\omega + \beta) U_3)^{p^s} \\ &= (1 - \omega^{p^t} (\omega + \beta)^{p^s}) U_1^{p^t} U_3^{p^s} + (1 - (\omega + \beta)^{p^s+p^t}) U_2^{p^t} U_3^{p^s} \\ &\quad - (1 - \omega^{p^t} (\omega + \beta)^{p^s}) U_1^{p^t} U_3^{p^s} - (1 - \omega^{p^s+p^t}) U_2^{p^t} U_3^{p^s}. \end{aligned}$$

Since $\omega^{p^t+p^s} = 1$ and $(\omega + \beta)^{p^t+p^s} = 1$, the right hand side of (3.2) is 0. This completes the proof. \square

Theorem 3.2. Fix $\omega, \beta \in \mathbb{F}_q$ with $\omega \neq 0, 1$. Let X be the corresponding Alexander f -quandle on \mathbb{F}_q . Then the set

$$\{U_1^{p^v} U_2^{p^u} \mid \omega^{p^v+p^u} = 1, (\omega + \beta)^{p^v+p^u} = 1; 0 \leq v < u < m\}$$

provides a basis of the second cohomology $H_Q^2((X, *, f); \mathbb{F}_q)$.

Example 3.3. Let p be an odd prime and v, u be non-negative integers. Let $\omega = -1$ and $\beta = 2$. Then we have $\omega^{p^v+p^u} = 1$ and $(\omega + \beta)^{p^v+p^u} = 1$. Hence, the set defined in Theorem 3.2 provides a basis for 2-cocycles.

Example 3.4. Let $f(x) = x^2 + x + 1 \in \mathbb{F}_2[x]$ and consider $\mathbb{F}_4 = \mathbb{F}_2[x]/(f)$. Let ω be a primitive element of \mathbb{F}_4 . Then the order of ω is 3. Let $\beta = \omega^2$. Note that $\omega^2 = \omega + 1$ and ω^2 is also a primitive element of \mathbb{F}_4 since it is a conjugate of ω with respect to \mathbb{F}_2 . We have

$$\omega^{2^0+2^1} = 1 \text{ and } (\omega + \beta)^{2^0+2^1} = 1,$$

Hence $\{U_1^{2^0} U_2^{2^1}\}$ provides a basis of the second cohomology $H_Q^2((X, *, f); \mathbb{F}_4)$.

4. THE 3-COCYCLES

In this section we give basis for the cohomology group $H_Q^3((X, *, f); \mathbb{F}_q)$.

For positive integers a and b , let

$$\mu_a(x, y) = (x + y)^a - x^a - y^a$$

and define

$$\psi(a, b) := (\mu_a(U_1, U_2) - \mu_a(\omega U_1, (\omega + \beta) U_2)) \cdot U_3^b.$$

Then we have the following proposition.

Proposition 4.1. If $\omega^{a+p^s} = 1$ and $(\omega + \beta)^{a+p^s} = 1$, then $\Psi(a, p^s)$ is a 3-cocycle.

Proof. Define

$$h(U_1, U_2) = \mu_a(U_1, U_2) - \mu_a(\omega U_1, (\omega + \beta) U_2).$$

Note that

$$\psi(a, b) := h(U_1, U_2) \cdot U_3^b.$$

Then by (2.5), we have

$$\delta(U_1^a) = (U_1 + U_2)^a - (\omega U_1 + (\omega + \beta) U_2)^a,$$

which implies

$$h(U_1, U_2) = \delta(U_1^a) - (1 - \omega^a) \cdot U_1^a - (1 - (\omega + \beta)^a) \cdot U_2^a.$$

Also, from (2.5), we have

$$\begin{aligned} & \delta(h(U_1, U_2)) \\ &= -h(U_1 + U_2, U_3) + h(\omega U_1 + (\omega + \beta) U_2, (\omega + \beta) U_3) + h(U_1, U_2 + U_3) \\ (4.1) \quad & - h(\omega U_1, \omega U_2 + (\omega + \beta) U_3) \\ &= (1 - \omega^a) h(U_1, U_2) - (1 - (\omega + \beta)^a) h(U_2, U_3) \\ &= (h(U_1, U_2) - h(U_2, U_3)) - (\omega^a h(U_1, U_2) - (\omega + \beta)^a h(U_2, U_3)). \end{aligned}$$

Since

$$\psi(a, b) = h(U_1, U_2) \cdot U_3^b,$$

from (2.6) and (4.1) we have

$$\begin{aligned} & \delta(\Psi(a, b)) \\ &= \delta(h(U_1, U_2)) \cdot U_4^b - h(U_1, U_2) \delta(U_3^b) \\ (4.2) \quad &= \left[(h(U_1, U_2) - h(U_2, U_3)) - (\omega + \beta)^b (\omega^a h(U_1, U_2) - (\omega + \beta)^a h(U_2, U_3)) \right] U_4^b \\ & - h(U_1, U_2) ((U_3 + U_4)^b - \omega^a (\omega U_3 + (\omega + \beta) U_4)^b). \end{aligned}$$

Let $b = p^s$. Then, from (4.2) we have

$$\begin{aligned} & \delta(\Psi(a, p^s)) \\ (4.3) \quad &= (1 - \omega^a (\omega + \beta)^{p^s}) h(U_1, U_2) U_4^{p^s} - (1 - (\omega + \beta)^{a+p^s}) h(U_2, U_3) U_4^{p^s} \\ & - (1 - \omega^{a+p^s}) h(U_1, U_2) U_3^{p^s} - (1 - \omega^a (\omega + \beta)^{p^s}) h(U_1, U_2) U_4^{p^s}. \end{aligned}$$

Since $\omega^{a+p^s} = 1$ and $(\omega + \beta)^{a+p^s} = 1$, the right hand side of (4.3) is 0. This completes the proof. □

Remark 4.2. Moreover, Ψ , defined above, is a coboundary; see [13].

Let $\chi(x, y) = \sum_{i=1}^{p-1} (-1)^{i-1} \cdot i^{-1} \cdot x^{p-i} \cdot y^i \equiv \frac{1}{p}((x+y)^p - x^p - y^p) \pmod{p}$.

Define

$$E_0(a \cdot p, b) = \left(\chi(U_1, U_2)^a - (\omega + \beta)^b \chi(\omega U_1, (\omega + \beta) U_2)^a \right) \cdot U_3^b.$$

Also, define

$$h(U_1, U_2) := \chi(U_1, U_2)^a - (\omega + \beta)^b \chi(\omega U_1, (\omega + \beta) U_2)^a.$$

Then we have

$$E_0(a \cdot p, b) = h(U_1, U_2) \cdot U_3^b.$$

Hence we have the following proposition.

Proposition 4.3. *If $\omega^{p^s+p^h} = 1$ and $(\omega + \beta)^{p^s+p^h} = 1$ with $s > 0$, then $E_0(p^s, p^h)$ is a 3-cocycle.*

Proof.

(4.4)

$$\begin{aligned} \delta(E_0(a \cdot p, b)) &= \delta(h(U_1, U_2)) \cdot U_4^b - h(U_1, U_2) \delta(U_3^b) \\ &= (1 - \omega^{ap} (\omega + \beta)^b) h(U_1, U_2) U_4^b - (1 - (\omega + \beta)^{ap+b}) h(U_2, U_3) U_4^b \\ &\quad - h(U_1, U_2) ((U_3 + U_4)^b - \omega^{ap} (\omega U_3 + (\omega + \beta) U_4)^b). \end{aligned}$$

Let $a = p^{s-1}$ and $b = p^h$. Then from equation (4.4) we have

$$\begin{aligned} \delta(E_0(p^s, p^h)) &= (1 - \omega^{p^s} (\omega + \beta)^{p^h}) h(U_1, U_2) \cdot U_4^{p^h} \\ &\quad - (1 - (\omega + \beta)^{p^s+p^h}) h(U_2, U_3) \cdot U_4^{p^h} \\ &\quad - (1 - \omega^{p^s+p^h}) U_3^{p^h} h(U_1, U_2) \\ &\quad - (1 - \omega^{p^s} (\omega + \beta)^{p^h}) h(U_1, U_2) \cdot U_4^{p^h}. \end{aligned} \tag{4.5}$$

Since $\omega^{p^s+p^h} = 1$ and $(\omega + \beta)^{p^s+p^h} = 1$, the right hand side of (4.5) is 0. This completes the proof. \square

Again, let $\chi(x, y) = \sum_{i=1}^{p-1} (-1)^{i-1} \cdot i^{-1} \cdot x^{p-i} \cdot y^i \equiv \frac{1}{p}((x+y)^p - x^p - y^p) \pmod{p}$.

Define

$$E_1(a, b \cdot p) = U_1^a \cdot \left(\chi(U_2, U_3)^b - \omega^a \chi(\omega U_2, (\omega + \beta) U_3)^b \right).$$

Also, define

$$h(U_2, U_3) := \chi(U_2, U_3)^b - \omega^a \chi(\omega U_2, (\omega + \beta) U_3)^b.$$

Then we have the following proposition.

Proposition 4.4. *If $\omega^{p^s+p^t} = 1$ and $(\omega + \beta)^{p^s+p^t} = 1$ with $s > 0$, then $E_1(p^t, p^s)$ is a 3-cocycle.*

Proof. Note that

$$E_1(a, b \cdot p) = U_1^a \cdot h(U_2, U_3).$$

We have

$$\begin{aligned}
 (4.6) \quad & \delta(E_1(a, b \cdot p)) \\
 &= \delta(U_1^a \cdot h(U_2, U_3)) \\
 &= \delta(U_1^a) h(U_3, U_4) - U_1^a \delta(h(U_2, U_3)) \\
 &= ((U_1 + U_2)^a - (\omega + \beta)^{p \cdot b} (\omega U_1 + (\omega + \beta) U_2)^a) h(U_3, U_4) \\
 &\quad - U_1^a (h(U_2 + U_3, U_4) - \omega^a h(\omega U_2 + (\omega + \beta) U_3, (\omega + \beta) U_4)) \\
 &\quad + U_1^a (h(U_2, U_3 + U_4) - \omega^a h(\omega U_2, \omega U_3 + (\omega + \beta) U_4)).
 \end{aligned}$$

Let $a = p^t$ and $b = p^{s-1}$. Then from (4.6) we have

$$\begin{aligned}
 (4.7) \quad & \delta(E_1(p^t, p^s)) \\
 &= U_1^{p^t} \cdot \left[(1 - \omega^{p^t} (\omega + \beta)^{p^s}) h(U_3, U_4) - h(U_2 + U_3, U_4) \right. \\
 &\quad \left. + \omega^{p^t} h(\omega U_2 + (\omega + \beta) U_3, (\omega + \beta) U_4) + h(U_2, U_3 + U_4) \right. \\
 &\quad \left. - \omega^{p^t} h(\omega U_2, \omega U_3 + (\omega + \beta) U_4) \right] + (1 - (\omega + \beta)^{p^t+p^s}) U_2^{p^t} h(U_3, U_4).
 \end{aligned}$$

Since $h(U_i, U_{i+1}) = \chi(U_i, U_{i+1})^{p^{s-1}} - \omega^{p^t} \chi(\omega U_i, (\omega + \beta) U_{i+1})^{p^{s-1}}$, $\omega^{p^s+p^t} = 1$, and $(\omega + \beta)^{p^s+p^t} = 1$, straightforward computation yields that the right hand side of (4.7) is 0. This completes the proof. \square

Let p be a prime, and v , u , and t be non-negative integers. Define $F(p^v, p^u, p^t) = U_1^{p^v} U_2^{p^u} U_3^{p^t} \in C^3$ where $p^v, p^u, p^t < q$.

Proposition 4.5. (1) *If $\omega^{p^v+p^u+p^t} = 1$ and $(\omega + \beta)^{p^v+p^u+p^t} = 1$, then $F(p^v, p^u, p^t)$ is a 3-cocycle.*

(2) *If $\omega^{p^v+p^u} = 1$ and $(\omega + \beta)^{p^v+p^u} = 1$, then $F(p^v, p^u, 0)$ is a 3-cocycle.*

Proof. We first prove (1).

$$\begin{aligned}
& \delta(F(p^v, p^u, p^t)) \\
&= \delta(U_1^{p^v} U_2^{p^u} U_3^{p^t}) \\
&= ((U_1 + U_2)^{p^v} - (\omega + \beta)^{p^u+p^t} (\omega U_1 + (\omega + \beta) U_2)^{p^v}) \cdot U_3^{p^u} \cdot U_4^{p^t} \\
&\quad - U_1^{p^v} \cdot ((U_2 + U_3)^{p^u} - \omega^{p^v} (\omega + \beta)^{p^t} (\omega U_2 + (\omega + \beta) U_3)^{p^u}) \cdot U_4^{p^t} \\
(4.8) \quad &+ U_1^{p^v} \cdot U_2^{p^u} ((U_3 + U_4)^{p^t} - \omega^{p^v+p^u} (\omega U_3 + (\omega + \beta) U_4)^{p^t}) \\
&= (1 - \omega^{p^v} (\omega + \beta)^{p^u+p^t}) U_1^{p^v} U_3^{p^u} U_4^{p^t} + (1 - (\omega + \beta)^{p^v+p^u+p^t}) U_2^{p^v} U_3^{p^u} U_4^{p^t} \\
&\quad - (1 - \omega^{p^v+p^u} (\omega + \beta)^{p^t}) U_1^{p^v} U_2^{p^u} U_4^{p^t} - (1 - \omega^{p^v} (\omega + \beta)^{p^t+p^u}) U_1^{p^v} U_3^{p^u} U_4^{p^t} \\
&\quad + (1 - \omega^{p^v+p^u+p^t}) U_1^{p^v} U_2^{p^u} U_3^{p^t} + (1 - \omega^{p^v+p^u} (\omega + \beta)^{p^t}) U_1^{p^v} U_2^{p^u} U_4^{p^t} \\
&= 0.
\end{aligned}$$

Since $\omega^{p^v+p^u+p^t} = 1$ and $(\omega + \beta)^{p^v+p^u+p^t} = 1$, the right hand side of (4.8) is 0.

In (2), by taking p^t as 0 in (4.8), and with $\omega^{p^v+p^u} = 1$ and $(\omega + \beta)^{p^v+p^u} = 1$, it can be shown in a similar manner that

$$\delta(F(p^v, p^u, 0)) = 0.$$

□

As in [13, 16], let Q be the set of all tuples (p^v, p^u, p^t, p^s) where p is a prime, such that $v < t, u < s, u \leq t$ and $\omega^{p^v+p^t} = \omega^{p^u+p^s} = (\omega + \beta)^{p^v+p^t} = (\omega + \beta)^{p^u+p^s} = 1$, and one of the following conditions hold.

Case I. $\omega^{p^v+p^u} = 1, (\omega + \beta)^{p^v+p^u} = 1$.

Case II. $\omega^{p^v+p^u} \neq 1, (\omega + \beta)^{p^v+p^u} \neq 1$ and $t > s$.

Case III. $\omega^{p^v+p^u} \neq 1, (\omega + \beta)^{p^v+p^u} \neq 1, t = s$ and $p \neq 2$.

Case IV. $\omega^{p^v+p^u} \neq 1, (\omega + \beta)^{p^v+p^u} \neq 1, u \leq v < t < s$ and $\omega^{p^v} = \omega^{p^u}, (\omega + \beta)^{p^v} = (\omega + \beta)^{p^u}$ when $p \neq 2$.

Case V. $\omega^{p^v+p^u} \neq 1, (\omega + \beta)^{p^v+p^u} \neq 1, u < v < t \leq s$ and $\omega^{p^v} = \omega^{p^u}, (\omega + \beta)^{p^v} = (\omega + \beta)^{p^u}$ when $p = 2$.

Moreover, if $p = 2$, we need $u < t$ as well.

For each $(p^v, p^u, p^t, p^s) \in Q$, we denote a cocycle by Γ .

Then we have the following proposition discussing case I. $\omega^{p^v+p^u} = 1, (\omega + \beta)^{p^v+p^u} = 1$.

Proposition 4.6. $\Gamma(p^v, p^u, p^t, p^s) = F(p^v, p^u + p^t, p^s)$ is a 3-cocycle.

Proof.

$$\begin{aligned}
& \delta(F(p^v, p^u + p^t, p^s)) \\
&= \delta(U_1^{p^v} U_2^{p^u+p^t} U_3^{p^s}) \\
&= \delta(U_1^{p^v}) U_3^{p^u+p^t} U_4^{p^s} - U_1^{p^v} \delta(U_3^{p^u+p^t}) U_4^{p^s} + U_1^{p^v} U_3^{p^u+p^t} \delta(U_4^{p^s}) \\
(4.9) \quad &= ((U_1 + U_2)^{p^v} - (\omega + \beta)^{p^u+p^t+p^s} (\omega U_1 + (\omega + \beta) U_2)^{p^v} U_3^{p^u+p^t} U_4^{p^s} \\
&\quad - U_1^{p^v} \cdot ((U_2 + U_3)^{p^u+p^t} - \omega^{p^v} (\omega + \beta)^{p^s} (\omega + \beta)^{p^v} (\omega U_2 + (\omega + \beta) U_3)^{p^u+p^t})) U_4^{p^s} \\
&\quad + U_1^{p^v} \cdot U_2^{p^u+p^t} \cdot ((U_3 + U_4)^{p^s} - \omega^{p^v+p^u+p^t} (\omega + \beta)^{p^v+p^u+p^t} (\omega U_3 + (\omega + \beta) U_4)^{p^s})
\end{aligned}$$

Note that $(x + y)^{p^u+p^t} = (x^{p^u} + y^{p^u})(x^{p^t} + y^{p^t})$, this reduced to

$$\begin{aligned}
(4.10) \quad & \delta(U_1^{p^v} U_2^{p^u+p^t} U_3^{p^s}) \\
&= (1 - \omega^{p^v} (\omega + \beta)^{p^u+p^t+p^s}) U_1^{p^v} U_3^{p^u+p^t} U_4^{p^s} + (1 - (\omega + \beta)^{p^u+p^t+p^s+p^v}) U_2^{p^v} U_3^{p^u+p^t} U_4^{p^s} \\
&\quad - (1 - \omega^{p^v+p^u+p^t} (\omega + \beta)^{p^s}) U_1^{p^v} U_2^{p^u+p^t} U_4^{p^s} - (1 - \omega^{p^v} (\omega + \beta)^{p^s+p^u+p^t}) U_1^{p^v} U_3^{p^u+p^t} U_4^{p^s} \\
&\quad - (1 - \omega^{p^v+p^u} (\omega + \beta)^{p^s+p^t}) U_1^{p^v} U_2^{p^u} U_3^{p^t} U_4^{p^s} - (1 - \omega^{p^v+p^t} (\omega + \beta)^{p^s+p^u}) U_1^{p^v} U_2^{p^t} U_3^{p^u} U_4^{p^s} \\
&\quad + (1 - \omega^{p^v+p^u+p^t+p^s}) U_1^{p^v} U_2^{p^u+p^t} U_3^{p^s} + (1 - \omega^{p^v+p^u+p^t} (\omega + \beta)^{p^s}) U_1^{p^v} U_2^{p^u+p^t} U_4^{p^s} \\
&= 0.
\end{aligned}$$

□

Then we have the following proposition discussing case II, $\omega^{p^v+p^u} \neq 1$, $(\omega + \beta)^{p^v+p^u} \neq 1$ and $t > s$.

Proposition 4.7. $\Gamma(p^v, p^u, p^t, p^s) = F(p^v, p^u + p^t, p^s) - F(p^u, p^v + p^s, p^t) - (\omega^{p^u} (\omega + \beta)^{p^s} - 1)^{-1} (1 - \omega^{p^u+p^v} (\omega + \beta)^{p^t+p^s}) F(p^v, p^u, p^t + p^s) + F(p^v + p^u, p^s, p^t)$ is a 3-cocycle.

Proof.

$$\begin{aligned}
(4.11) \quad & \delta(F(p^v, p^u + p^t, p^s)) - \delta(F(p^u, p^v + p^s, p^t)) \\
&= (\omega^{p^u} (\omega + \beta)^{p^s} - 1)^{-1} (1 - \omega^{p^u+p^v} (\omega + \beta)^{p^t+p^s}) \delta(F(p^v, p^u, p^t + p^s) + \delta(F(p^v + p^u, p^s, p^t))) \\
&= -(1 - \omega^{p^v+p^u} (\omega + \beta)^{p^s+p^t}) U_1^{p^v} U_2^{p^u} U_3^{p^t} U_4^{p^s} + (1 - \omega^{p^v+p^u} (\omega + \beta)^{p^s+p^t}) U_1^{p^u} U_2^{p^v} U_3^{p^s} U_4^{p^t} \\
&\quad - (\omega^{p^u} (\omega + \beta)^{p^s} - 1)^{-1} (1 - \omega^{p^u+p^v} (\omega + \beta)^{p^t+p^s}) [(1 - \omega^{p^v+p^u+p^t} (\omega + \beta)^{p^s}) U_1^{p^v} U_2^{p^u} U_3^{p^t} U_4^{p^s} \\
&\quad - (1 - \omega^{p^u} (\omega + \beta)^{p^s+p^t+p^v}) U_1^{p^u} U_2^{p^v} U_3^{p^s} U_4^{p^t}] \\
&= 0.
\end{aligned}$$

□

Then we have the following proposition discussing case III, $\omega^{p^v+p^u} \neq 1$, $(\omega + \beta)^{p^v+p^u} \neq 1$, $t = s$ and $p \neq 2$. In [21], it is shown we can present this case as follows:

Proposition 4.8. $\Gamma(p^v, p^u, p^t, p^s) = F(p^v, p^t + p^s, p^u)$ is a 3-cocycle.

Proof. The proof is similar to that of Proposition 4.6.

□

Then we have the following proposition discussing cases IV and V, $\omega^{p^v+p^u} \neq 1$, $(\omega + \beta)^{p^v+p^u} \neq 1$, $u \leq v < t < s$ and $\omega^{p^v} = \omega^{p^u}$, $(\omega + \beta)^{p^v} = (\omega + \beta)^{p^u}$ when $p \neq 2$ and $\omega^{p^v+p^u} \neq 1$, $(\omega + \beta)^{p^v+p^u} \neq 1$, $u < v < t \leq s$ and $\omega^{p^v} = \omega^{p^u}$, $(\omega + \beta)^{p^v} = (\omega + \beta)^{p^u}$ when $p = 2$. In [21], it is shown we can present this case as follows:

Proposition 4.9. $\Gamma(p^v, p^u, p^t, p^s) = F(p^t, p^v + p^u, p^s)$ is a 3-cocycle.

Proof. The proof is similar to that of Proposition 4.6. \square

Theorem 4.10. Fix $\omega, \beta \in \mathbb{F}_q$ with $\omega \neq 0, \pm 1$. Let X be the corresponding Alexander f -quandle on \mathbb{F}_q where $H_Q^2((X, *, f); \mathbb{F}_q) \cong 0$. Then the set

$$\begin{aligned} I = & \{F(p^v, p^u, p^t) \mid \omega^{p^v+p^u+p^t} = (\omega + \beta)^{p^v+p^u+p^t} = 1, p^v < p^u < p^t < q\} \\ & \cup \{F(p^v, p^u, 0) \mid \omega^{p^v+p^u} = (\omega + \beta)^{p^v+p^u} = 1, p^v < p^u < q\} \\ & \cup \{E_0(p \cdot p^v, p^u) \mid \omega^{p^{v+1}+p^u} = (\omega + \beta)^{p^{v+1}+p^u} = 1, p^v < p^u < q\} \\ & \cup \{E_1(p^v, p \cdot p^u) \mid \omega^{p^v+p^{u+1}} = (\omega + \beta)^{p^v+p^{u+1}} = 1, p^v \leq p^u < q\} \\ & \cup \{\Gamma(p^v, p^u, p^t, p^s) \mid (p^v, p^u, p^t, p^s) \in Q(q)\} \end{aligned}$$

provides a basis of the third cohomology $H_Q^3((X, *, f); \mathbb{F}_q)$.

Example 4.11. Let p be an odd prime and v, u and t be non-negative integers. Let $\omega = -1$ and $\beta = 2$. Hence, $\omega^{p^v+p^u+p^t} \neq 1$ and $(\omega + \beta)^{p^v+p^u+p^t} = 1$. Then we have the following.

- (1) $F(p^v, p^u, p^t)$ is not a 3-cocycle.
- (2) $F(p^v, p^u, 0)$ is a 3-cocycle since $\omega^{p^v+p^u} = 1$ and $(\omega + \beta)^{p^v+p^u} = 1$. Also, $E_0(p^{v+1}, p^u)$ and $E_1(p^v, p^{u+1})$ are 3-cocycles.

Moreover,

$$Q(q) = \{(p^v, p^u, p^t, p^s) \mid p^u \leq p^t, p^v < p^t, p^u < p^s\}, \text{ and } \omega^{p^v+p^u} = (\omega + \beta)^{p^v+p^u} = 1 \text{ for any } (p^v, p^u, p^t, p^s) \in Q(q).$$

Therefore,

$$\begin{aligned} & \{F(p^v, p^u, 0) \mid 0 < p^v < p^u < q\} \cup \{E_0(p^{v+1}, p^u) \mid p^v < p^u < q\} \cup \{E_1(p^v, p^{u+1}) \mid p^v < p^u < q\} \\ & \cup \{F(p^v, p^u + p^t, p^s) \mid p^u \leq p^t, p^v < p^t, p^u < p^s, p^i < q, \text{ for all } i \in \{v, u, t, s\}\} \end{aligned}$$

is a basis for the cohomology group $H_Q^3((X, *, f); \mathbb{F}_q)$.

Remark 4.12. Example 4.11 shows that when $\beta = 1 - \omega$, the basis for the cohomology group $H_Q^3((X, *, f); \mathbb{F}_q)$ above is the same as the basis for the cohomology group $H_Q^3((X, *); \mathbb{F}_q)$, explained in [16, Subsection 2.4.1].

Example 4.13. Let $f(x) = x^3 + x^2 + 1 \in \mathbb{F}_2[x]$ and consider $\mathbb{F}_8 = \mathbb{F}_2[x]/(f)$. Let ω be a primitive element of \mathbb{F}_8 . Then the order of ω is 7. Let $\beta = \omega^{2^2}$. Note that $\omega^{2^2} = \omega^3 + \omega$ and ω^{2^2} is also a primitive element of \mathbb{F}_8 since it is a conjugate of ω with respect to \mathbb{F}_2 . We have

$$\omega^{2^0+2^1+2^2} = 1 \text{ and } (\omega + \beta)^{2^0+2^1+2^2} = 1,$$

but $\omega^{2^i+2^j} \neq 1$ for $i, j \in \{0, 1, 2\}$. Hence $H_Q^3((X, *, f); \mathbb{F}_8)$ is generated by $\{F(2^0, 2^1, 2^2)\}$.

5. THE 4-COCYCLES

In this section, we give some propositions showing some particular polynomials are 4-cocycles. The main theorem gives basis for the cohomology group $H_Q^4((X, *, f); \mathbb{F}_q)$ under the condition that the group $H_Q^2((X, *, f); \mathbb{F}_q)$ is trivial.

Proposition 5.1. *If $\omega^{p^v+p^u+p^t+p^s} = 1$ and $(\omega + \beta)^{p^v+p^u+p^t+p^s} = 1$, then the polynomial $U_1^{p^v} U_2^{p^u} U_3^{p^t} U_4^{p^s}$ is a 4-cocycle.*

Proof.

(5.1)

$$\begin{aligned}
& \delta(U_1^{p^v} U_2^{p^u} U_3^{p^t} U_4^{p^s}) \\
&= ((U_1 + U_2)^{p^v} - (\omega + \beta)^{p^u+p^t+p^s} (\omega U_1 + (\omega + \beta) U_2)^{p^v}) U_3^{p^u} U_4^{p^t} U_5^{p^s} \\
&- U_1^{p^v} \cdot ((U_2 + U_3)^{p^u} - \omega^{p^v} (\omega + \beta)^{p^t+p^s} (\omega U_2 + (\omega + \beta) U_3)^{p^u}) U_4^{p^t} U_5^{p^s} \\
&+ U_1^{p^v} \cdot U_2^{p^u} \cdot ((U_3 + U_4)^{p^t} - \omega^{p^v+p^u} (\omega + \beta)^{p^s} (\omega U_3 + (\omega + \beta) U_4)^{p^t}) U_5^{p^s} \\
&- U_1^{p^v} \cdot U_2^{p^u} \cdot U_3^{p^t} \cdot ((U_4 + U_5)^{p^s} - \omega^{p^v+p^u+p^t} (\omega U_4 + (\omega + \beta) U_5)^{p^s}) \\
&= (1 - \omega^{p^v} (\omega + \beta)^{p^u+p^t+p^s}) U_1^{p^v} U_3^{p^u} U_4^{p^t} U_5^{p^s} + (1 - (\omega + \beta)^{p^v+p^u+p^t+p^s}) U_2^{p^v} U_3^{p^u} U_4^{p^t} U_5^{p^s} \\
&- (1 - \omega^{p^v+p^u} (\omega + \beta)^{p^t+p^s}) U_1^{p^v} U_2^{p^u} U_4^{p^t} U_5^{p^s} - (1 - \omega^{p^v} (\omega + \beta)^{p^u+p^t+p^s}) U_1^{p^v} U_3^{p^u} U_4^{p^t} U_5^{p^s} \\
&+ (1 - \omega^{p^v+p^u+p^t} (\omega + \beta)^{p^s}) U_1^{p^v} U_2^{p^u} U_3^{p^t} U_5^{p^s} + (1 - \omega^{p^v+p^u} (\omega + \beta)^{p^s+p^t}) U_1^{p^v} U_2^{p^u} U_4^{p^t} U_5^{p^s} \\
&- (1 - \omega^{p^v+p^u+p^t+p^s}) U_1^{p^v} U_2^{p^u} U_3^{p^t} U_4^{p^s} - (1 - \omega^{p^v+p^u+p^t} (\omega + \beta)^{p^s}) U_1^{p^v} U_2^{p^u} U_3^{p^t} U_5^{p^s}.
\end{aligned}$$

Since $\omega^{p^v+p^u+p^t+p^s} = 1$ and $(\omega + \beta)^{p^v+p^u+p^t+p^s} = 1$, the right hand side of (5.1) is 0. This completes the proof. \square

We recall $\chi(x, y) = \sum_{i=1}^{p-1} (-1)^{i-1} \cdot i^{-1} \cdot x^{p-i} \cdot y^i \equiv \frac{1}{p} ((x+y)^p - x^p - y^p) \pmod{p}$.

Proposition 5.2. *If $\omega^{p^{u+1}+p^t+p^s} = 1$ and $(\omega + \beta)^{p^{u+1}+p^t+p^s} = 1$, then the polynomial $(\chi(U_1, U_2)^{p^u} - (\omega + \beta)^{p^t+p^s} \chi(\omega U_1, (\omega + \beta) U_2)^{p^u}) U_3^{p^t} U_4^{p^s}$ is a 4-cocycle.*

Proof. Let $h(U_1, U_2) = \chi(U_1, U_2)^{p^u} - (\omega + \beta)^{p^t+p^s} \chi(\omega U_1, (\omega + \beta) U_2)^{p^u}$.

We show that $h(U_1, U_2) U_3^{p^t} U_4^{p^s}$ is a 4-cocycle.

(5.2)

$$\begin{aligned}
& \delta(h(U_1, U_2) U_3^{p^t} U_4^{p^s}) \\
&= \delta(h(U_1, U_2) U_4^{p^t} U_5^{p^s}) \\
&- h(U_1, U_2) ((U_3 + U_4)^{p^t} - \omega^{p^{u+1}} (\omega + \beta)^{p^s} (\omega U_3 + (\omega + \beta) U_4)^{p^t}) U_5^{p^s} \\
&+ h(U_1, U_2) U_3^{p^t} ((U_4 + U_5)^{p^s} - \omega^{p^{u+1}+p^t} (\omega U_4 + (\omega + \beta) U_5)^{p^s}) \\
&= (1 - \omega^{p^{u+1}} (\omega + \beta)^{p^s+p^t}) h(U_1, U_2) U_4^{p^t} U_5^{p^s} - (1 - (\omega + \beta)^{p^{u+1}+p^t+p^s}) h(U_2, U_3) U_4^{p^t} U_5^{p^s} \\
&- (1 - \omega^{p^{u+1}+p^t} (\omega + \beta)^{p^s}) h(U_1, U_2) U_3^{p^t} U_5^{p^s} - (1 - \omega^{p^{u+1}} (\omega + \beta)^{p^s+p^t}) h(U_1, U_2) U_4^{p^t} U_5^{p^s} \\
&+ (1 - \omega^{p^{u+1}+p^t+p^s}) h(U_1, U_2) U_3^{p^t} U_4^{p^s} + (1 - \omega^{p^{u+1}+p^t} (\omega + \beta)^{p^s}) h(U_1, U_2) U_3^{p^t} U_5^{p^s}.
\end{aligned}$$

Since $\omega^{p^{u+1}+p^t+p^s} = 1$ and $(\omega + \beta)^{p^{u+1}+p^t+p^s} = 1$, the right hand side of (5.2) is 0. This completes the proof. \square

Proposition 5.3. *If $\omega^{p^v+p^{t+1}+p^s} = 1$ and $(\omega + \beta)^{p^v+p^{t+1}+p^s} = 1$, then the polynomial $U_1^{p^v} (\chi(U_2, U_3)^{p^t} - \omega^{p^v} (\omega + \beta)^{p^s} \chi(\omega U_2, (\omega + \beta) U_3)^{p^t}) U_4^{p^s}$ is a 4-cocycle.*

Proof. Let $h(U_2, U_3) = \chi(U_2, U_3)^{p^t} - \omega^{p^v} (\omega + \beta)^{p^s} \chi(\omega U_2, (\omega + \beta) U_3)^{p^t}$.

Now we show that $U_1^{p^v} h(U_2, U_3) U_4^{p^s}$ is a 4-cocycle.

(5.3)

$$\begin{aligned}
& \delta(U_1^{p^v} h(U_2, U_3) U_4^{p^s}) \\
&= ((U_1 + U_2)^{p^v} - (\omega + \beta)^{p^{t+1}+p^s} (\omega U_1 + (\omega + \beta) U_2)^{p^v}) h(U_3, U_4) U_5^{p^s} \\
&- U_1^{p^v} (h(U_2 + U_3, U_4) - \omega^{p^v} (\omega + \beta)^{p^s} h(\omega U_2 + (\omega + \beta) U_3, (\omega + \beta) U_4)) U_5^{p^s} \\
&+ U_1^{p^v} (h(U_2, U_3 + U_4) - \omega^{p^v} (\omega + \beta)^{p^s} h(\omega U_2, \omega U_3 + (\omega + \beta) U_4)) U_5^{p^s} \\
&- U_1^{p^v} h(U_2, U_3) ((U_4 + U_5)^{p^s} - \omega^{p^v+p^{t+1}} (\omega U_4 + (\omega + \beta) U_5)^{p^s}) \\
&= U_1^{p^v} \left[(1 - \omega^{p^v} (\omega + \beta)^{p^{t+1}+p^s}) h(U_3, U_4) - h(U_2 + U_3, U_4) + h(U_2, U_3 + U_4) \right. \\
&+ \omega^{p^v} (\omega + \beta)^{p^s} h(\omega U_2 + (\omega + \beta) U_3, (\omega + \beta) U_4) \\
&- \omega^{p^v} (\omega + \beta)^{p^s} h(\omega U_2, \omega U_3 + (\omega + \beta) U_4) - (1 - \omega^{p^v+p^{t+1}} (\omega + \beta)^{p^s}) h(U_2, U_3) \left. \right] U_5^{p^s} \\
&+ (1 - (\omega + \beta)^{p^v+p^{t+1}+p^s}) U_2^{p^v} h(U_3, U_4) U_5^{p^s} - (1 - \omega^{p^v+p^{t+1}+p^s}) U_1^{p^v} h(U_2, U_3) U_4^{p^s}.
\end{aligned}$$

Since

$$h(U_i, U_{i+1}) = \chi(U_i, U_{i+1})^{p^t} - \omega^{p^v} (\omega + \beta)^{p^s} \chi(\omega U_i, (\omega + \beta) U_{i+1})^{p^t},$$

and $\omega^{p^v+p^{t+1}+p^s} = 1$ and $(\omega + \beta)^{p^v+p^{t+1}+p^s} = 1$, it can be shown that the right hand side of (5.3) is 0. This completes the proof. \square

Proposition 5.4. *If $\omega^{p^v+p^u+p^{s+1}} = 1$ and $(\omega + \beta)^{p^v+p^u+p^{s+1}} = 1$, then the polynomial*

$$U_1^{p^v} U_2^{p^u} \left(\chi(U_3, U_4)^{p^s} - \omega^{p^v+p^u} \chi(\omega U_3, (\omega + \beta) U_4)^{p^s} \right)$$

is a 4-cocycle.

Proof. Let $h(U_3, U_4) = \chi(U_3, U_4)^{p^s} - \omega^{p^v+p^u} \chi(\omega U_3, (\omega + \beta) U_4)^{p^s}$.

Now we claim that $U_1^{p^v} U_2^{p^u} h(U_3, U_4)$ is a 4-cocycle.

$$\begin{aligned}
 (5.4) \quad & \delta(U_1^{p^v} U_2^{p^u} h(U_3, U_4)) \\
 &= \delta(U_1^{p^v}) U_3^{p^u} h(U_4, U_5) - U_1^{p^v} \delta(U_2^{p^u}) h(U_4, U_5) + U_1^{p^v} U_2^{p^u} \delta(h(U_3, U_4)) \\
 &= ((U_1 + U_2)^{p^v} - (\omega + \beta)^{p^u+p^{s+1}} (\omega U_1 + (\omega + \beta) U_2)^{p^v}) U_3^{p^u} h(U_4, U_5) \\
 &\quad - U_1^{p^v} ((U_2 + U_3)^{p^u} - \omega^{p^v} (\omega + \beta)^{p^{s+1}} (\omega U_2 + (\omega + \beta) U_3)^{p^u}) h(U_4, U_5) \\
 &\quad + U_1^{p^v} U_2^{p^u} \left(h(U_3 + U_4, U_5) - \omega^{p^v+p^u} h(\omega U_3 + (\omega + \beta) U_4, (\omega + \beta) U_5) \right) \\
 &\quad - U_1^{p^v} U_2^{p^u} \left(h(U_3, U_4 + U_5) - \omega^{p^v+p^u} h(\omega U_3, \omega U_4 + (\omega + \beta) U_5) \right) \\
 &= (1 - (\omega + \beta)^{p^v+p^u+p^{s+1}}) U_2^{p^v} U_3^{p^u} h(U_4, U_5) + U_1^{p^v} U_2^{p^u} \left[h(U_3 + U_4, U_5) - h(U_3, U_4 + U_5) \right. \\
 &\quad \left. - (1 - \omega^{p^v+p^u} (\omega + \beta)^{p^{s+1}}) h(U_4, U_5) + \omega^{p^v+p^u} h(\omega U_3, \omega U_4 + (\omega + \beta) U_5) \right. \\
 &\quad \left. - \omega^{p^v+p^u} h(\omega U_3 + (\omega + \beta) U_4, (\omega + \beta) U_5) \right]
 \end{aligned}$$

Since

$$h(U_i, U_{i+1}) = \chi(U_i, U_{i+1})^{p^s} - \omega^{p^v+p^u} \chi(\omega U_i, (\omega + \beta) U_{i+1})^{p^s},$$

$\omega^{p^v+p^u+p^{s+1}} = 1$, and $(\omega + \beta)^{p^v+p^u+p^{s+1}} = 1$, it can be shown that the right hand side of (5.4) is 0. This completes the proof. \square

Proposition 5.5. *If $\omega^{p^i+p^j+p^u+p^t+p^s} = 1$, $(\omega + \beta)^{p^i+p^j+p^u+p^t+p^s} = 1$, $\omega^{p^i+p^j} = \omega^{p^i+p^u} = 1$ and $(\omega + \beta)^{p^u+p^t+p^s} = (\omega + \beta)^{p^j+p^t+p^s} = 1$, then the polynomial $U_1^{p^i} U_2^{p^j+p^u} U_3^{p^t} U_4^{p^s}$ is a 4-cocycle.*

Proof.

$$\begin{aligned}
 (5.5) \quad & \delta(U_1^{p^i} U_2^{p^j+p^u} U_3^{p^t} U_4^{p^s}) \\
 &= ((U_1 + U_2)^{p^i} - (\omega + \beta)^{p^j+p^u+p^t+p^s} (\omega U_1 + (\omega + \beta) U_2)^{p^i}) U_3^{p^j+p^u} U_4^{p^t} U_5^{p^s} \\
 &\quad - U_1^{p^i} \cdot ((U_2 + U_3)^{p^j+p^u} - \omega^{p^i} (\omega + \beta)^{p^t+p^s} (\omega U_2 + (\omega + \beta) U_3)^{p^j+p^u}) U_4^{p^t} U_5^{p^s} \\
 &\quad + U_1^{p^i} \cdot U_2^{p^j+p^u} \cdot ((U_3 + U_4)^{p^t} - \omega^{p^i+p^j+p^u} (\omega + \beta)^{p^s} (\omega U_3 + (\omega + \beta) U_4)^{p^t}) U_5^{p^s} \\
 &\quad - U_1^{p^i} \cdot U_2^{p^j+p^u} \cdot U_3^{p^t} \cdot ((U_4 + U_5)^{p^s} - \omega^{p^i+p^j+p^u+p^t} (\omega U_4 + (\omega + \beta) U_5)^{p^s})
 \end{aligned}$$

Note that $(x + y)^{p^j+p^u} = (x^{p^j} + y^{p^j})(x^{p^u} + y^{p^u})$. Hence, from (5.5) we have

$$\begin{aligned}
& \delta(U_1^{p^i} U_2^{p^j+p^u} U_3^{p^t} U_4^{p^s}) \\
&= (1 - \omega^{p^i} (\omega + \beta)^{p^j+p^u+p^t+p^s}) U_1^{p^i} U_3^{p^j+p^u} U_4^{p^t} U_5^{p^s} \\
&+ (1 - (\omega + \beta)^{p^i+p^j+p^u+p^t+p^s}) U_2^{p^i} U_3^{p^j+p^u} U_4^{p^t} U_5^{p^s} \\
&- (1 - \omega^{p^i+p^j+p^u} (\omega + \beta)^{p^t+p^s}) U_1^{p^i} U_2^{p^j+p^u} U_4^{p^t} U_5^{p^s} \\
&- (1 - \omega^{p^i} (\omega + \beta)^{p^j+p^u+p^t+p^s}) U_1^{p^i} U_3^{p^j+p^u} U_4^{p^t} U_5^{p^s} \\
&- (1 - \omega^{p^i+p^j} (\omega + \beta)^{p^u+p^t+p^s}) U_1^{p^i} U_2^{p^j} U_3^{p^u} U_4^{p^t} U_5^{p^s} \\
&- (1 - \omega^{p^i+p^u} (\omega + \beta)^{p^j+p^t+p^s}) U_1^{p^i} U_2^{p^u} U_3^{p^j} U_4^{p^t} U_5^{p^s} \\
&+ (1 - \omega^{p^i+p^j+p^u+p^t} (\omega + \beta)^{p^s}) U_1^{p^i} U_2^{p^j+p^u} U_3^{p^t} U_5^{p^s} \\
&+ (1 - \omega^{p^i+p^j+p^u} (\omega + \beta)^{p^s+p^t}) U_1^{p^i} U_2^{p^j+p^u} U_4^{p^t} U_5^{p^s} \\
&- (1 - \omega^{p^i+p^j+p^u+p^t+p^s}) U_1^{p^i} U_2^{p^j+p^u} U_3^{p^t} U_4^{p^s} \\
&- (1 - \omega^{p^i+p^j+p^u+p^t} (\omega + \beta)^{p^s}) U_1^{p^i} U_2^{p^j+p^u} U_3^{p^t} U_5^{p^s}.
\end{aligned}
\tag{5.6}$$

Since $\omega^{p^i+p^j+p^u+p^t+p^s} = 1$, $(\omega + \beta)^{p^i+p^j+p^u+p^t+p^s} = 1$, $\omega^{p^i+p^j} = \omega^{p^i+p^u} = 1$ and $(\omega + \beta)^{p^u+p^t+p^s} = (\omega + \beta)^{p^j+p^t+p^s} = 1$, the right hand side of (5.6) is 0. This completes the proof. \square

Proposition 5.6. *If $\omega^{p^{u+1}+p^{s+1}} = 1$ and $(\omega + \beta)^{p^{u+1}+p^{s+1}} = 1$, then the polynomial $(\chi(U_1, U_2)^{p^u} - (\omega + \beta)^{p^{s+1}} \chi(\omega U_1, (\omega + \beta) U_2)^{p^u}) (\chi(U_3, U_4)^{p^s} - \omega^{p^{u+1}} \chi(\omega U_3, (\omega + \beta) U_4)^{p^s})$ is a 4-cocycle.*

Proof. Let

$$h(U_1, U_2) = \chi(U_1, U_2)^{p^u} - (\omega + \beta)^{p^{s+1}} \chi(\omega U_1, (\omega + \beta) U_2)^{p^u}$$

and

$$h^*(U_3, U_4) = \chi(U_3, U_4)^{p^s} - \omega^{p^{u+1}} \chi(\omega U_3, (\omega + \beta) U_4)^{p^s}.$$

We claim that $h(U_1, U_2) h^*(U_3, U_4)$ is a 4-cocycle.

$$\begin{aligned}
& \delta(h(U_1, U_2) h^*(U_3, U_4)) \\
&= \delta(h(U_1, U_2)) h^*(U_4, U_5) - h(U_1, U_2) \delta(h^*(U_3, U_4)) \\
&= h(U_1, U_2) \left[h^*(U_3, U_4 + U_5) - h^*(U_3 + U_4, U_5) + (1 - \omega^{p^{u+1}} (\omega + \beta)^{p^{s+1}}) h^*(U_4, U_5) \right. \\
&\quad \left. - \omega^{p^{u+1}} h^*(\omega U_3, \omega U_4 + (\omega + \beta) U_5) + \omega^{p^{u+1}} h^*(\omega U_3 + (\omega + \beta) U_4, (\omega + \beta) U_5) \right] \\
&\quad - (1 - (\omega + \beta)^{p^{u+1}+p^{s+1}}) h(U_2, U_3) h^*(U_4, U_5)
\end{aligned}
\tag{5.7}$$

Since $h(U_i, U_{i+1}) = \chi(U_i, U_{i+1})^{p^u} - (\omega + \beta)^{p^{s+1}} \chi(\omega U_i, (\omega + \beta) U_{i+1})^{p^u}$,

$h^*(U_i, U_{i+1}) = \chi(U_i, U_{i+1})^{p^s} - \omega^{p^{u+1}} \chi(\omega U_i, (\omega + \beta) U_{i+1})^{p^s}$, $\omega^{p^{u+1}+p^{s+1}} = 1$, and $(\omega + \beta)^{p^{u+1}+p^{s+1}} = 1$, it can be shown that the right hand side of (5.7) is 0. This completes the proof. \square

Proposition 5.7. *If $\omega^{p^i+p^j+p^u+p^{s+1}} = 1$, $(\omega + \beta)^{p^i+p^j+p^u+p^{s+1}} = 1$, $\omega^{p^i+p^j} = \omega^{p^i+p^u} = 1$, and $(\omega + \beta)^{p^u+p^{s+1}} = (\omega + \beta)^{p^j+p^{s+1}} = 1$, then the polynomial*

$$U_1^{p^i} U_2^{p^j+p^u} \left(\chi(U_3, U_4)^{p^s} - \omega^{p^i+p^j+p^u} \chi(\omega U_3, (\omega + \beta) U_4)^{p^s} \right)$$

is a 4-cocycle.

Proof. Let $h(U_3, U_4) = \chi(U_3, U_4)^{p^s} - \omega^{p^i+p^j+p^u} \chi(\omega U_3, (\omega + \beta) U_4)^{p^s}$. We claim that $U_1^{p^i} U_2^{p^j+p^u} h(U_3, U_4)$ is a 4-cocycle.

$$\begin{aligned}
 & \delta(U_1^{p^i} U_2^{p^j+p^u} h(U_3, U_4)) \\
 &= \delta(U_1^{p^i}) U_3^{p^j+p^u} h(U_4, U_5) - U_1^{p^i} \delta(U_2^{p^j+p^u}) h(U_4, U_5) + U_1^{p^i} U_2^{p^j+p^u} \delta(h(U_3, U_4)) \\
 &= ((U_1 + U_2)^{p^i} - (\omega + \beta)^{p^j+p^u+p^{s+1}} (\omega U_1 + (\omega + \beta) U_2)^{p^i}) U_3^{p^j+p^u} h(U_4, U_5) \\
 &\quad - U_1^{p^i} \cdot ((U_2 + U_3)^{p^j+p^u} - \omega^{p^i} (\omega + \beta)^{p^{s+1}} (\omega U_2 + (\omega + \beta) U_3)^{p^j+p^u}) h(U_4, U_5) \\
 &\quad + U_1^{p^i} U_2^{p^j+p^u} (h(U_3 + U_4, U_5) - \omega^{p^i+p^j+p^u} h(\omega U_3 + (\omega + \beta) U_4, (\omega + \beta) U_5)) \\
 &\quad - U_1^{p^i} U_2^{p^j+p^u} (h(U_3, U_4 + U_5) - \omega^{p^i+p^j+p^u} h(\omega U_3, \omega U_4 + (\omega + \beta) U_5)) \\
 &= (1 - \omega^{p^i} (\omega + \beta)^{p^j+p^u+p^{s+1}}) U_1^{p^i} U_3^{p^j+p^u} h(U_4, U_5) \\
 &\quad + (1 - (\omega + \beta)^{p^i+p^j+p^u+p^{s+1}}) U_2^{p^i} U_3^{p^j+p^u} h(U_4, U_5) \\
 &\quad - (1 - \omega^{p^i+p^j+p^u} (\omega + \beta)^{p^{s+1}}) U_1^{p^i} U_2^{p^j+p^u} h(U_4, U_5) \\
 &\quad - (1 - \omega^{p^i+p^j} (\omega + \beta)^{p^u+p^{s+1}}) U_1^{p^i} U_2^{p^j} U_3^{p^u} h(U_4, U_5) \\
 &\quad - (1 - \omega^{p^i+p^u} (\omega + \beta)^{p^j+p^{s+1}}) U_1^{p^i} U_2^{p^u} U_3^{p^j} h(U_4, U_5) \\
 &\quad - (1 - \omega^{p^i} (\omega + \beta)^{p^j+p^u+p^{s+1}}) U_1^{p^i} U_3^{p^j+p^u} h(U_4, U_5) \\
 &\quad + U_1^{p^i} U_2^{p^j+p^u} (h(U_3 + U_4, U_5) - \omega^{p^i+p^j+p^u} h(\omega U_3 + (\omega + \beta) U_4, (\omega + \beta) U_5)) \\
 &\quad - U_1^{p^i} U_2^{p^j+p^u} (h(U_3, U_4 + U_5) - \omega^{p^i+p^j+p^u} h(\omega U_3, \omega U_4 + (\omega + \beta) U_5))
 \end{aligned} \tag{5.8}$$

Since $h(U_i, U_{i+1}) = \chi(U_i, U_{i+1})^{p^s} - \omega^{p^i+p^j+p^u} \chi(\omega U_i, (\omega + \beta) U_{i+1})^{p^s}$, $\omega^{p^i+p^j+p^u+p^{s+1}} = 1$, $(\omega + \beta)^{p^i+p^j+p^u+p^{s+1}} = 1$, $\omega^{p^i+p^j} = \omega^{p^i+p^u} = 1$, and $(\omega + \beta)^{p^u+p^{s+1}} = (\omega + \beta)^{p^j+p^{s+1}} = 1$, the right hand side of (5.8) is 0. This completes the proof. \square

Proposition 5.8. *If $\omega^{p^i+p^j+p^v+p^u+p^t+p^s} = 1$, $(\omega + \beta)^{p^i+p^j+p^v+p^u+p^t+p^s} = 1$, $\omega^{p^i+p^j} = \omega^{p^i+p^v} = \omega^{p^v+p^u} = \omega^{p^v+p^t} = 1$, and $(\omega + \beta)^{p^s+p^t} = (\omega + \beta)^{p^s+p^u} = (\omega + \beta)^{p^v+p^u} = (\omega + \beta)^{p^j+p^u} = 1$, then the polynomial $U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u+p^t} U_4^{p^s}$ is a 4-cocycle.*

Proof.

(5.9)

$$\begin{aligned}
& \delta(U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u+p^t} U_4^{p^s}) \\
&= ((U_1 + U_2)^{p^i} - (\omega + \beta)^{p^j+p^v+p^u+p^t+p^s} (\omega U_1 + (\omega + \beta)U_2)^{p^i}) U_3^{p^j+p^v} U_4^{p^u+p^t} U_5^{p^s} \\
&- U_1^{p^i} \cdot ((U_2 + U_3)^{p^j+p^v} - \omega^{p^i} (\omega + \beta)^{p^u+p^t+p^s} (\omega U_2 + (\omega + \beta)U_3)^{p^j+p^v}) U_4^{p^u+p^t} U_5^{p^s} \\
&+ U_1^{p^i} \cdot U_2^{p^j+p^v} \cdot ((U_3 + U_4)^{p^u+p^t} - \omega^{p^i+p^j+p^v} (\omega + \beta)^{p^s} (\omega U_3 + (\omega + \beta)U_4)^{p^u+p^t}) U_5^{p^s} \\
&- U_1^{p^i} \cdot U_2^{p^j+p^v} \cdot U_3^{p^u+p^t} \cdot ((U_4 + U_5)^{p^s} - \omega^{p^i+p^j+p^v+p^u+p^t} (\omega U_4 + (\omega + \beta)U_5)^{p^s}).
\end{aligned}$$

Note that $(x + y)^{p^j+p^u} = (x^{p^j} + y^{p^j})(x^{p^u} + y^{p^u})$. Hence, from (5.9) we have

$$\begin{aligned}
& \delta(U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u+p^t} U_4^{p^s}) \\
&= (1 - \omega^{p^i} (\omega + \beta)^{p^j+p^v+p^u+p^t+p^s}) U_1^{p^i} U_3^{p^j+p^v} U_4^{p^u+p^t} U_5^{p^s} \\
&+ (1 - (\omega + \beta)^{p^i+p^j+p^v+p^u+p^t+p^s}) U_2^{p^i} U_3^{p^j+p^v} U_4^{p^u+p^t} U_5^{p^s} \\
&- (1 - \omega^{p^i+p^j+p^v} (\omega + \beta)^{p^u+p^t+p^s}) U_1^{p^i} U_2^{p^j+p^v} U_4^{p^u+p^t} U_5^{p^s} \\
&- (1 - \omega^{p^i} (\omega + \beta)^{p^j+p^v+p^u+p^t+p^s}) U_1^{p^i} U_3^{p^j+p^v} U_4^{p^u+p^t} U_5^{p^s} \\
&- (1 - \omega^{p^i+p^j} (\omega + \beta)^{p^v+p^u+p^t+p^s}) U_1^{p^i} U_2^{p^j} U_3^{p^v} U_4^{p^u+p^t} U_5^{p^s} \\
(5.10) \quad & - (1 - \omega^{p^i+p^v} (\omega + \beta)^{p^j+p^u+p^t+p^s}) U_1^{p^i} U_2^{p^j} U_3^{p^v} U_4^{p^u+p^t} U_5^{p^s} \\
&+ (1 - \omega^{p^i+p^j+p^v+p^u+p^t} (\omega + \beta)^{p^s}) U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u+p^t} U_5^{p^s} \\
&+ (1 - \omega^{p^i+p^j+p^v} (\omega + \beta)^{p^u+p^t+p^s}) U_1^{p^i} U_2^{p^j+p^v} U_4^{p^u+p^t} U_5^{p^s} \\
&+ (1 - \omega^{p^i+p^j+p^v+p^u} (\omega + \beta)^{p^s+p^t}) U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u} U_4^{p^t} U_5^{p^s} \\
&+ (1 - \omega^{p^i+p^j+p^v+p^t} (\omega + \beta)^{p^s+p^u}) U_1^{p^i} U_2^{p^j+p^v} U_3^{p^t} U_4^{p^u} U_5^{p^s} \\
&- (1 - \omega^{p^i+p^j+p^v+p^u+p^t+p^s}) U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u+p^t} U_4^{p^s} \\
&- (1 - \omega^{p^i+p^j+p^v+p^u+p^t} (\omega + \beta)^{p^s}) U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u+p^t} U_5^{p^s}.
\end{aligned}$$

Since $\omega^{p^i+p^j+p^v+p^u+p^t+p^s} = 1$, $(\omega + \beta)^{p^i+p^j+p^v+p^u+p^t+p^s} = 1$, $\omega^{p^i+p^j} = \omega^{p^i+p^v} = \omega^{p^v+p^u} = \omega^{p^v+p^t} = 1$, and $(\omega + \beta)^{p^s+p^t} = (\omega + \beta)^{p^s+p^u} = (\omega + \beta)^{p^v+p^u} = (\omega + \beta)^{p^j+p^u} = 1$, the right hand side of (5.10) is 0. This completes the proof. \square

Let $q = p^m$, where m is a positive integer.

$$A = \left\{ U_1^{p^v} U_2^{p^u} U_3^{p^t} U_4^{p^s} \mid \omega^{p^v+p^u+p^t+p^s} = 1, (\omega + \beta)^{p^v+p^u+p^t+p^s} = 1, 0 \leq v < u < t < s < m \right\},$$

$$B = \left\{ (\chi(U_1, U_2)^{p^u} - (\omega + \beta)^{p^t+p^s} \chi(\omega U_1, (\omega + \beta) U_2)^{p^u}) U_3^{p^t} U_4^{p^s} \mid \omega^{p^{u+1}+p^t+p^s} = 1, (\omega + \beta)^{p^{u+1}+p^t+p^s} = 1, 0 \leq u < t < s < m \right\},$$

$$C = \left\{ U_1^{p^v} (\chi(U_2, U_3)^{p^t} - \omega^{p^v} (\omega + \beta)^{p^s} \chi(\omega U_2, (\omega + \beta) U_3)^{p^t}) U_4^{p^s} \mid \omega^{p^v+p^{t+1}+p^s} = 1, (\omega + \beta)^{p^v+p^{t+1}+p^s} = 1, 0 \leq v \leq t < s < m \right\},$$

$$D = \left\{ U_1^{p^v} U_2^{p^u} (\chi(U_3, U_4)^{p^s} - \omega^{p^v+p^u} \chi(\omega U_3, (\omega + \beta) U_4)^{p^s}) \mid \omega^{p^v+p^u+p^{s+1}} = 1, (\omega + \beta)^{p^v+p^u+p^{s+1}} = 1, 0 \leq v < u \leq s < m \right\}, \text{ and}$$

$$E = \Gamma(p^v, p^u, p^t, 0) = \left\{ U_1^{p^v} U_2^{p^u} U_3^{p^t} \mid \omega^{p^v+p^u+p^t} = 1, (\omega + \beta)^{p^v+p^u+p^t} = 1, 0 \leq v < u < t < m \right\}.$$

Then we have the following theorem.

Theorem 5.9. *Fix $\omega, \beta \in \mathbb{F}_q$ with $\omega \neq 0, \pm 1$. Let X be the corresponding Alexander f -quandle on \mathbb{F}_q where $H_Q^2((X, *, f); \mathbb{F}_q) \cong 0$. Then the set $A \cup B \cup C \cup D \cup E$ provides a basis of the fourth cohomology $H_Q^4((X, *, f); \mathbb{F}_q)$.*

Example 5.10. *Let $f(x) = x^4 + x + 1 \in \mathbb{F}_2[x]$ and consider $\mathbb{F}_{16} = \mathbb{F}_2[x]/(f)$. Let ω be a primitive element of \mathbb{F}_{16} . Then the order of ω is 15. Let $\beta = \omega^{2^2}$. Note that $\omega^{2^2} = \omega + 1$ and ω^{2^2} is also a primitive element of \mathbb{F}_{16} since it is a conjugate of ω with respect to \mathbb{F}_2 . We have*

$$\omega^{2^0+2^1+2^2+2^3} = 1 \text{ and } (\omega + \beta)^{2^0+2^1+2^2+2^3} = 1,$$

*but $\omega^{2^i+2^j+2^k} \neq 1$ for $i, j, k \in \{0, 1, 2, 3\}$. Hence $H_Q^4((X, *, f); \mathbb{F}_{16})$ is generated by A .*

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REFERENCES

- [1] K. Ameer, M. Saito, *Polynomial cocycles of Alexander quandles and applications*, J. Knot Theory Ramifications 18 (2009), no. 2, 151165.
- [2] S. Abe, *On 4-cocycles of Alexander quandles on finite fields*, J. Knot Theory Ramifications 23 (2014), no. 8, 1450043, 30 pp.
- [3] J. S. Carter; M. Elhamdadi; M. Granña; M. Saito, *Cocycle knot invariants from quandle modules and generalized quandle homology*. Osaka J. Math. 42 (2005), no. 3, 49-541.
- [4] J. S. Carter; D. Jelsovsky; S. Kamada; L. Langford; M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*. Trans. Amer. Math. Soc. 355 (2003), no. 10, 3947-3989.
- [5] W. E. Clark; M. Elhamdadi; M. Saito; T. Yeatman, *Quandle colorings of knots and applications*. J. Knot Theory Ramifications 23 (2014), no. 6, 1450035, 29 pp.
- [6] I. R. U. Churchill, M. Elhamdadi, M. Green, A. Makhlof, *f -racks, f -quandles, their extensions and cohomology*, to appear in J. Algebra Appl., 2017.
- [7] M. Elhamdadi; E. Moutuou, *Foundations of topological racks and quandles*. J. Knot Theory Ramifications 25 (2016), no. 3, 1640002, 17 pp.
- [8] M. Elhamdadi; S. Nelson, *Quandles—an introduction to the algebra of knots*. Student Mathematical Library, 74. American Mathematical Society, Providence, RI, 2015. x+245 pp.
- [9] X. Hou, *Automorphism groups of Alexander quandles*. J. Algebra 344 (2011), 373-385.
- [10] X. Hou, *Finite modules over $\mathbb{Z}[t, t^{-1}]$* . J. Knot Theory Ramifications 21 (2012), no. 8, 1250079, 28 pp.

- [11] A. Hulpke; D. Stanovsky; P. Vojtechovsky, *Connected quandles and transitive groups*. J. Pure Appl. Algebra 220 (2016), no. 2, 735-758.
- [12] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Alg., 23, 37-65.
- [13] J. Mandemaker, *Various Topics in Rack and Quandle Homology*, Master's Thesis, Radboud University Nijmegen (2010).
- [14] Makhlouf, A., Silvestrov, S., *Hom-algebras and Hom-coalgebras*, J. Algebra Appl. 9 (2010), no. 4, 553–589.
- [15] S. Matveev, *Distributive groupoids in knot theory*, (Russian) Mat. Sb. (N.S.) 119(161) (1982), no. 1, 78–88, 160.
- [16] T. Mochizuki, *The 3-cocycles of the Alexander quandles $\mathbb{F}_q[T]/(T - \omega)$* , Algebr. Geom. Topol. 5 (2005), 183205.
- [17] T. Mochizuki, *Some calculations of cohomology groups of finite Alexander quandles*, J. Pure Appl. Algebra (2003), 287-330.
- [18] S. Nelson, *The combinatorial revolution in knot theory*. Notices Amer. Math. Soc. 58 (2011), no. 11, 1553-1561.
- [19] M. Niebrzydowski, J. H. Przytycki, *Homology operations on homology of quandles*. J. of Algebra (2010), 1529-1548.
- [20] T. Nosaka, *On quandle homology groups of Alexander quandles of prime order*, Trans. Amer. Math. Soc. 365 (2013), no. 7, 3413-3436.
- [21] T. Nosaka, *On third homologies of groups and of quandles via the Dijkgraaf-Witten invariant and Inoue-Kabaya map*, Algebr. Geom. Topol., (2014), no. 5, 2655–2691.
- [22] R. Rubinsztein, *Topological quandles and invariants of links*. J. Knot Theory Ramifications 16 (2007), no. 6, 789-808.
- [23] N. Takahashi, *Quandle varieties, generalized symmetric spaces, and ϕ -spaces*. Transform. Groups 21 (2016), no. 2, 555-576.

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